Free Dirac evolution as a quantum random walk

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Abstract

Any positive-energy state of a free Dirac particle that is initially highly-localized, evolves in time by spreading at speeds close to the speed of light. This general phenomenon is explained by the fact that the Dirac evolution can be approximated arbitrarily closely by a quantum random walk, where the roles of coin and walker systems are naturally attributed to the spin and position degrees of freedom of the particle. Initially entangled and spatially localized spin-position states evolve with asymptotic two-horned distributions of the position probability, familiar from earlier studies of quantum walks. For the Dirac particle, the two horns travel apart at close to the speed of light.

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1 Introduction

The concept of a quantum random walk (QRW) has been widely discussed and extended in various directions [1] since its introduction [2, 3, 4] and developement [5, 6, 7, 8, 9, 10, 11, 12]. Much of the interest has derived from an expectation that such a mathematically attractive idea should have important applications in quantum information theory, analogous to known applications of classical random walks (CRWs) in classical information science.

On the other hand, CRWs also have many applications outside classical information theory, in a wide variety of areas of science where mathematical modelling is involved, so it should not be surprising if QRWs find applications outside quantum information theory. Here we describe such an application to relativistic quantum mechanics; for an earlier application in this field, see [13].

The evolution in time of the state of a free Dirac particle, starting from a highly localized, positive-energy state, is a quantum process that has only recently been described fully [14]. There had been a widespread misapprehension that no relativistic particle with nonzero rest-mass m could be localized much within its Compton wavelength $\lambda_C = \hbar/mc$, where c is the speed of light. However it has now been shown [15] that there is no such difficulty for the Dirac particle if localization is characterized in terms of the Dirac position operator $\mathbf{x}$, by making $\Delta_x = \sqrt{\langle \mathbf{x}^2 \rangle - \langle \mathbf{x} \rangle^2}$ small while keeping the energy positive, and not by unrealistic attempts to restrict the domain of the wavefunction to a bounded region in configuration space. Arbitrarily precise localization, with $\Delta_x \ll \lambda_C$, is possible in the case of the free Dirac particle with positive energy. When the particle is localized in such an initial state, it has an associated uncertainty in energy $\Delta_E \gg mc^2$, and the subsequent evolution produces a probability density that spreads outwards in all directions at close to speed c. The graph of the evolving density along any axis through the centre of initial localization (see Fig. 1 in [14]) shows a striking resemblance to the two-horned density found for a typical 1-dimensional QRW [4]. For the Dirac particle, the horns are close to distance ct from the starting point. We shall show that this is not a coincidence, and that the evolution of *any* positive-energy state of a free Dirac particle moving in 1-dimension can be modelled arbitrarily closely as a QRW of the type described in detail by Ambainis *et al.* [4], Konno [16, 17] and others.

In addition to providing a somewhat surprising application of a QRW to a real process, this result provides some new insights as to the nature of the quantum walk itself. Until now the various proposed realization schemes for QRWs were typically based on the idea that the coin and walker degrees of

freedom of the walk should be associated with two distinct quantum systems. These two systems were to be combined by means of some form of dynamical coupling-decoupling scheme. The present application shows that alternatively, a single quantum mechanical object such as the free Dirac particle — by its very nature as a relativistic system with translational and spin degrees of freedom — can be identified in the course of its time evolution with a quantum random walk. This natural occurence of a QRW, instead of some engineered realization, suggests that the question of its ontological status is still an interesting and open one.

The present application draws attention to two other important features of QRWs that have been emphasized by others [6, 18]. The first is that a QRW is a unitary evolution; the associated randomness is of the kind associated with every unitary quantum evolution. In particular, a QRW is typically reversible in time, unlike a CRW. The second feature is that a QRW is typically a ballistic process, associated with spreading at a constant speed, unlike diffusive CRWs, where spreading is proportional to the square root of the time. These two features are essential for the application that we describe below to the free Dirac evolution, which is a time-reversible process characterized by spreading near the speed of light.

In what follows, we relate the evolution of Dirac's equation to that of a QRW based on the canonical Heisenberg algebra extended by the Dirac matrices. Then we construct and discuss the limiting probability distribution describing the translational spreading of an initial state. The paper concludes with some speculations about the physical reality of the quantum walk of the Dirac particle, and the possibility of detecting it experimentally.

2 QRW and free Dirac evolution

The free Dirac Hamiltonian operator for a particle with zero momentum along the y and z directions is

$$H(\hat{p}) = c\alpha\hat{p} + mc^2\beta\,, \quad \hat{p} = -i\hbar\, d/dx\,, \tag{1}$$

acting on 4-component spinor wavefunctions $\Psi(x)$. Here we adopt a representation of the Dirac matrices with

$$\alpha = \sigma_3 \otimes \sigma_3\,, \quad \beta = \sigma_2 \otimes \mathbf{1}_2\,, \tag{2}$$

where σ_i, $i = 1, 2, 3$ are the usual Pauli matrices, and $\mathbf{1}_2$ is the 2×2 unit matrix. In this representation the helicity (spin) operator associated with rotations about the x-axis is $\Sigma = \mathbf{1}_2 \otimes \sigma_3$. From this point onwards we adopt

the natural units $\hbar = c = m = 1$. Recalling that only those solutions of Dirac's equation with positive energy describe physical electron states, we introduce the orthonormal positive-energy spinors in momentum space

$$u_\pm(p) = \frac{1}{2\sqrt{E(p)(E(p)+1)}} \begin{pmatrix} 1 + E(p) \pm p \\ \\ i(1 + E(p) \mp p) \end{pmatrix} \otimes e_\pm , \tag{3}$$

where $e_+ = (1,0)^T$, $e_- = (0,1)^T$ and $E(p) = \sqrt{p^2+1}$. These spinors satisfy the relations

$$u_\pm(p)^\dagger u_\pm(p) = 1\,, \quad u_\pm(p)^\dagger u_\mp(p) = 0\,,$$

$$H(p)u_\pm(p) = E(p)u_\pm(p)\,, \quad \Sigma u_\pm(p) = \pm\frac{1}{2}u_\pm(p)\,. \tag{4}$$

Now we can write an arbitrary positive-energy wavefunction (with zero y and z components of momentum) $\Psi_0(x)$ in terms of two arbitrary functions $f_\pm(p)$ as

$$\Psi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \left\{ f_+(p)u_+(p) + f_-(p)u_-(p) \right\} dp\,. \tag{5}$$

Then

$$\int_{-\infty}^{\infty} \Psi_0(x)^\dagger \Psi_0(x)\, dx = 1 \longleftrightarrow \int_{-\infty}^{\infty} \left\{ |f_+(p)|^2 + |f_-(p)|^2 \right\} dp = 1\,. \tag{6}$$

Suppose now that we choose a normalized positive-energy state with a definite helicity $+1/2$ and finite mean energy. Then

$$f_+(p) = f(p)\,, \quad f_-(p) = 0\,, \quad \int_{-\infty}^{\infty} |f(p)|^2\, dp = 1, \tag{7}$$

and

$$\langle\, H(\hat{p}\,)\, \rangle = \int_{-\infty}^{\infty} E(p)|f(p)|^2\, dp = E_0 < \infty\,. \tag{8}$$

With $f_-(p) = 0$, the action of $H(\,\hat{p}\,)$ in the second factor of the tensor product space in (2) and (3) becomes trivial, as the second spinor remains constant at the value e_+. Thus the second factor space can be ignored, and we can consider $H(\,\hat{p}\,)$ to have the *effective* form

$$H(\,\hat{p}\,) = \sigma_3 \hat{p} + \sigma_2 \tag{9}$$

acting in the first factor space. In this first space, we write

$$(1 \quad 0)^T = |+\rangle \quad (0 \quad 1)^T = |-\rangle\,, \tag{10}$$

so that the positive-energy spinor $u_+(p)$ in (3) takes the (effective) form

$$u_+(p) = \frac{1}{2\sqrt{E(p)(E(p)+1)}}\left\{(1+E(p)+p)|+\rangle + i(1+E(p)-p)|-\rangle\right\}\,. \tag{11}$$

Next we consider a fixed, small time interval $\Delta t \ll 1/E_0$. The (effective) unitary evolution operator for the Dirac particle can then be approximated over the time interval Δt using the relations

$$\begin{aligned} e^{-iH(\hat{p})\Delta t} &= V\,U + O([\Delta t]^2)\,, \\ V &= e^{-i\Delta t\sigma_3\hat{p}}\,, \quad U = e^{-i\Delta t\sigma_2}\,. \end{aligned} \tag{12}$$

Here we see the appearance of the evolution operator $V\,U$ for a 1-dimensional QRW [4], with V enacting a step of length Δt to the left or right along the x-axis (the "walker space"), depending on the sign of σ_3, and with the reshuffling matrix U representing the "quantum coin toss" after each time interval of duration Δt. For a longer time $t = n\Delta t$, we have from (12)

$$e^{-iH(\hat{p})t} = (V\,U)^n + \mathrm{O}(\Delta t)\,, \tag{13}$$

and we see that the evolution of the state of the Dirac particle over any finite time t can be obtained arbitrarily accurately by replacing the exact evolution operator by $(V\,U)^n$ and letting $n \to \infty$ and $\Delta t \to 0$ with $n\Delta t = t$. In other words,

$$\lim_{\substack{n\to\infty \\ \Delta t\to 0 \\ n\Delta t = t}} \left(e^{-i\Delta t\sigma_3\hat{p}}\, e^{-i\Delta t\sigma_2}\right)^n = e^{-iH(\hat{p})t}\,, \tag{14}$$

and we can emulate the Dirac evolution by the evolution of a QRW. If we are interested in particular in the evolution of the probability density along the x-axis for the electron, we need only go at each time t to the asymptotic form of the QRW probability distribution for the "walker" [4, 16, 17].

It is important to note at this point that whereas the exact Dirac evolution operator $e^{-iH(\hat{p})t}$ obviously preserves the positive energy condition for physically meaningful states, the same is not true of the approximate, QRW

evolution $(V\,U)^n$. However, (13) and (14) show that in the asymptotic limit described, the positive energy condition is respected.

We close this section with the following remark. Rewriting the evolution operators as $V = |+\rangle\langle+|\, e^{-i\Delta t\hat{p}} + |-\rangle\langle-|\, e^{i\Delta t\hat{p}}$, $U = e^{-i\Delta t\sigma_2}$, we identify the type of quantum walk involved here as a Canonical Algebra QRW (CA-QRW) in the classification of [19]. In contrast to the Euclidean QRW, which takes place on the integers and whose evolution operator is constructed from the generators of the Euclidean algebra [19, 20, 21, 22], in the present case the generators of the canonical Heisenberg algebra — position and momentum operators — are used in the construction of a discrete walk on the x-coordinate axis. The close algebraic relationship between these two walks facilitates the solution of the time evolution in the present case, provided (as is done in next section) that we carefully discretize the coordinate-space (generalized) eigenfunctions, which unlike their Euclidean QRW counterparts, are not orthogonal.

3 Asymptotic solutions and localization

Let $\mathcal{H}$ denote the Hilbert space spanned by all vectors $|\Phi\rangle \otimes |\pm\rangle$, corresponding in the coordinate representation to normalizable 2-component wavefunctions $\Phi(x)|\pm\rangle$. Introduce a dense subspace $\mathcal{S} < \mathcal{H}$ consisting of all finite linear combinations of suitably regular vectors $|\Phi\rangle \otimes |\pm\rangle \in \mathcal{H}$, say all those corresponding to $\Phi(x) = P(x)\, e^{-\alpha x^2}$, where $P(x)$ is an arbitrary polynomial and α is some fixed positive constant. Then denote by $\mathcal{S}^*$ the space dual to $\mathcal{S}$ and, with the usual abuse of notation, consider $\mathcal{H}$ as a subspace of $\mathcal{S}^*$, so that we obtain the Gelf'and Triple (or Rigged Hilbert Space [23])

$$\mathcal{S} < \mathcal{H} < \mathcal{S}^* \,. \tag{15}$$

The space $\mathcal{S}^*$ contains in particular the vectors $|x'\rangle \otimes |\pm\rangle$, where $|x'\rangle$ is the generalized eigenvector of the Dirac x-coordinate operator $\hat{q}$,

$$\hat{q}|x'\rangle = x'|x'\rangle \,, \tag{16}$$

corresponding in the coordinate representation to $\delta(x - x')$.

The introduction of the time interval Δt as in (12), in turn defines a length interval Δt on the x-axis (recall that $c = 1$ now), and a corresponding direct-integral decomposition

$$\mathcal{S}^* = \oplus \int_{-\Delta t/2}^{\Delta t/2} \mathcal{V}_{x_0}\, dx_0 \,, \tag{17}$$

where $\mathcal{V}_{x_0} < \mathcal{S}^*$ is spanned by all vectors of the form $|x_0 + m\Delta t\rangle \otimes |\pm\rangle$, with $x_0 \in (-\Delta t/2,\, \Delta t/2]$ fixed, and $m \in \mathbb{Z}$. We note at once that each $\mathcal{V}_{x_0}$ is invariant under the action of the QRW evolution operator $V\,U$, *i.e.*

$$V\,U\,\mathcal{V}_{x_0} < \mathcal{V}_{x_0}\,, \tag{18}$$

because

$$V\,|x_0 + m\Delta t\rangle \otimes |\pm\rangle = |x_0 + (m \mp 1)\Delta t\rangle \otimes |\pm\rangle\,. \tag{19}$$

In order to describe the QRW evolution more fully, we now write the initial state with wave function as in (5) and (7), as an entangled state of the walker and coin subsystems,

$$|\Psi_0\rangle\!\rangle = \int_{-\infty}^{\infty} \{c_+(x)|x\rangle \otimes |+\rangle + c_-(x)|x\rangle \otimes |-\rangle\}\;dx\,, \tag{20}$$

where

$$\begin{aligned} c_+(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1 + E(p) + p}{2\sqrt{E(p)(E(p)+1)}} f(p)e^{ipx}dp\,, \\ c_-(x) &= \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{1 + E(p) - p}{2\sqrt{E(p)(E(p)+1)}} f(p)e^{ipx}dp\,. \end{aligned} \tag{21}$$

Normalization of $|\Psi_0\rangle\!\rangle$ is satisfied because

$$\int_{-\infty}^{\infty} \left\{|c_+(x)|^2 + |c_-(x)|^2\right\}\;dx = 1, \tag{22}$$

as a consequence of (4) and (6). At this point we emphasize again that although, as is well known [24], $|x\rangle \otimes |\pm\rangle$ is *not* a positive-energy (generalized) state, $|\Psi_0\rangle$ *is* a positive-energy state, as a consequence of the particular form of the coefficients in (21).

The expansion in (20) can be rewritten as

$$\begin{aligned} |\Psi_0\rangle\!\rangle \;=\; & \sum_{m\in\mathbb{Z}}\int_{-\Delta t/2}^{\Delta t/2} \{c_+(x_0 + m\Delta t)\,|x_0 + m\Delta t\rangle \otimes |+\rangle \\ & + c_-(x_0 + m\Delta t)\,|x_0 + m\Delta t\rangle \otimes |-\rangle\}\;dx_0\,, \end{aligned} \tag{23}$$

which is to be compared with (17). In view of the invariance of each $\mathcal{V}_{x_0}$ under the action of the QRW evolution, we can restrict our attention to that action on each substate

$$\begin{aligned} |\Phi_{x_0}\rangle\!\rangle \;=\; & \sum_{m\in\mathbb{Z}} \{c_+(x_0 + m\Delta t)\,|x_0 + m\Delta t\rangle \otimes |+\rangle \\ & + c_-(x_0 + m\Delta t)\,|x_0 + m\Delta t\rangle \otimes |-\rangle\}\,\sqrt{\Delta t} \end{aligned} \tag{24}$$

with x_0 fixed, even though these substates are not normalizable, and are not positive-energy states. The point is that the general form of any such substate is preserved under the action of the QRW evolution $V\,U$, with no change in the value of x_0. The inclusion of the multiplicative factor $\sqrt{\Delta t}$ in last equation is for later convenience with the normalization.

Consider firstly the action of V on a general substate $|\Phi_0\rangle\!\rangle \in \mathcal{V}_{x_0}$, say one with $x_0 = 0$ for definiteness. We have

$$\begin{aligned} V|\Phi_0\rangle\!\rangle &= \sum_{m\in\mathbb{Z}} |m\Delta t\rangle \otimes \{c_+((m-1)\Delta t)|+\rangle + c_-((m+1)\Delta t)|-\rangle\}\sqrt{\Delta t} \\ &= \sum_{m\in\mathbb{Z}}\sum_{\alpha=\pm}[c_\alpha(m\Delta t)\,|(m+\alpha)\Delta t\rangle \otimes |\alpha\rangle]\sqrt{\Delta t} \\ &\equiv (E_+ \otimes P_+ + E_- \otimes P_-)\ |\Phi_0\rangle\!\rangle\,, \end{aligned} \tag{25}$$

where

$$E_\pm|m\Delta t\rangle = |(m\pm 1)\Delta t\rangle\,, \quad P_\pm|\pm\rangle = |\pm\rangle\,, \quad P_\mp|\pm\rangle = 0\,. \tag{26}$$

The action of U on $|\Phi_0\rangle\!\rangle$ is easily seen from (12), which implies that

$$\begin{aligned} U\,|+\rangle &= \cos(\Delta t)\,|+\rangle + \sin(\Delta t)\,|-\rangle\,, \\ U\,|-\rangle &= \cos(\Delta t)\,|-\rangle - \sin(\Delta t)\,|+\rangle\,. \end{aligned} \tag{27}$$

Combining (25) and (27), we see that

$$V\,U\,|\Phi_0\rangle\!\rangle = (E_+ \otimes P_+U + E_- \otimes P_-U)\,|\Phi_0\rangle\!\rangle\,. \tag{28}$$

If we had taken $f_+(p) = 0$, $f_-(p) = f(p)$ in (7), we would have written instead

$$u_-(p) = \frac{1+E(p)-p}{2\sqrt{E(p)(E(p)+1)}}|+\rangle + i\frac{1+E(p)+p}{2\sqrt{E(p)(E(p)+1)}}|-\rangle\,, \tag{29}$$

and we would have obtained

$$|\Psi_0\rangle\!\rangle = \int_{-\infty}^{\infty} \{c_+(x)|x\rangle \otimes |+\rangle + c_-(x)|x\rangle \otimes |-\rangle\}\ dx, \tag{30}$$

where now

$$\begin{aligned} c_+(x) &= \frac{1}{\sqrt{2\pi}}\int \frac{1+E(p)-p}{2\sqrt{E(p)(E(p)+1)}} f(p)e^{ipx}dp\,, \\ c_-(x) &= \frac{i}{\sqrt{2\pi}}\int \frac{1+E(p)+p}{2\sqrt{E(p)(E(p)+1)}} f(p)e^{ipx}dp\,. \end{aligned}$$

Then, decomposing $|\Psi_0\rangle\rangle$ into substates $|\Phi_{x_0}\rangle\rangle$ as before, we would have obtained on a state of this general form, say one with $x_0 = 0$, that

$$V\,U|\Phi_0\rangle\rangle = \{E_- \otimes P_+ U + E_+ \otimes P_- U\}\,|\Phi_0\rangle\rangle\,. \tag{31}$$

We will treat here the first case, as the second one can be treated similarly.

To proceed we choose $-\pi \leq \phi < \pi$, and set

$$|\phi/\Delta t\rangle = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{-im\phi}|m\Delta t\rangle\,, \tag{32}$$

so that

$$E_\pm\,|\phi/\Delta t\rangle = e^{\pm i\phi}|\phi/\Delta t\rangle\,, \qquad |m\Delta t\rangle = \int_{-\pi}^{\pi} e^{im\phi}|\phi/\Delta t\rangle\,d\phi\,. \tag{33}$$

Considering the evolution operator $V\,U$ acting as in (31), but now with $E_\pm$ diagonalized, we have

$$V\,U(\phi) = \left(e^{i\phi}P_+ + e^{-i\phi}P_-\right)U\,. \tag{34}$$

The eigenvalues of this 2×2 matrix with parameter ϕ are

$$\lambda_\pm(\phi) = \cos\phi\cos\Delta t \pm i\sqrt{1-\cos^2\phi\cos^2\Delta t}\,. \tag{35}$$

Suppose that the corresponding eigenvectors are

$$\begin{aligned} |v_+(\phi)\rangle &= f_{++}(\phi)|+\rangle + f_{+-}(\phi)|-\rangle\,, \\ |v_-(\phi)\rangle &= f_{-+}(\phi)|+\rangle + f_{--}(\phi)|-\rangle\,. \end{aligned} \tag{36}$$

Then the eigenvectors of $V\,U$ are of the form $|\phi/\Delta t\rangle \otimes |v_\pm(\phi)\rangle$, with eigenvalues $\lambda_\pm(\phi)$. Expanding $|\Phi_0\rangle\rangle$ in terms of these eigenvectors of $V\,U$ we get

$$|\Phi_0\rangle\rangle = \int_{-\pi}^{\pi}\{g_+(\phi)\,|\phi/\Delta t\rangle \otimes |v_+(\phi)\rangle + g_-(\phi)\,|\phi/\Delta t\rangle \otimes |v_-(\phi)\rangle\}\sqrt{\Delta t}\;d\phi\,, \tag{37}$$

where

$$g_\pm(\phi) = \sum_{m\in Z}\left\{c_+(m\Delta t)f^*_{\pm+}(\phi) + c_-(m\Delta t)f^*_{\pm-}(\phi)\right\}e^{im\phi}\,. \tag{38}$$

Hence

$$\begin{aligned} |\Phi_n\rangle\rangle \;&\equiv\; (V\,U)^n|\Phi_0\rangle\rangle = \int_{-\pi}^{\pi}\{g_+(\phi)\lambda_+(\phi)^n|\phi/\Delta t\rangle \otimes |v_+(\phi)\rangle \\ &\qquad + g_-(\phi)\lambda_-(\phi)^n|\phi/\Delta t\rangle \otimes |v_-(\phi)\rangle\}\sqrt{\Delta t}\;d\phi\,. \end{aligned} \tag{39}$$

If we now denote by X_n the random variable defining the "walker position" after n evolution steps, then we obtain for the "quantum statistical moment"

$$\begin{aligned}\langle (X_n)^k \rangle &\equiv \langle\!\langle \Phi_n | \hat{q}^k \otimes \mathbf{1} | \Phi_n \rangle\!\rangle = Tr_{S+T}\left(|\Phi_n\rangle\!\rangle \langle\!\langle \Phi_n | \hat{q}^k \otimes \mathbf{1} \right) \\ &= Tr_T\left((Tr_S |\Phi_n\rangle\!\rangle \langle\!\langle \Phi_n|) \hat{q}^k \right) = Tr_T\left(\rho_T^{(n)} \hat{q}^k \right), \end{aligned} \tag{40}$$

where the expectation value has been expressed in terms of traces over the translational degree of freedom (T) of the Dirac particle —the walker system in the parlance of QRW — and its spin (S) — the coin system for the QRW. This has allowed us to cast the "quantum statistical moment" in terms of the reduced density operator $\rho_T^{(n)} = (Tr_S |\Phi_n\rangle\!\rangle\langle\!\langle\Phi_n|)$ which, as it provides all the necessary statistical information about the position of the Dirac particle, could have also been the main object of our mathematical investigation, as happens in most studies of QRWs.

We proceed to determine the statistical moment of the translation operator, which takes the form

$$\begin{aligned}\langle (X_n)^k \rangle &= \int_{-\pi}^{\pi} \left[g_+^*(\phi)\lambda_+^*(\phi)^n (-i\partial_\phi)^k \left\{ g_+(\phi)\lambda_+(\phi)^n \right\} \right. \\ &\left. + g_-^*(\phi)\lambda_-^*(\phi)^n (-i\partial_\phi)^k \left\{ g_-(\phi)\lambda_-(\phi)^n \right\} \right] \frac{d\phi}{2\pi} (\Delta t)^{k+1}, \end{aligned} \tag{41}$$

or equivalently

$$\begin{aligned}\langle (X_n)^k \rangle &= (n\Delta t)^k \int_{-\pi}^{\pi} \left\{ |g_+(\phi)|^2 \left(-i\lambda'_+(\phi)/\lambda_+(\phi) \right)^k \right. \\ &\left. + |g_-(\phi)|^2 \left(-i\lambda'_-(\phi)/\lambda_-(\phi) \right)^k \right\} \frac{d\phi}{2\pi} \Delta t + \mathrm{O}(n\Delta t)^{k-1}. \end{aligned} \tag{42}$$

Hence as $n \to \infty$, $\Delta t \to 0$, with $t = n\Delta t$ large, we have that

$$\begin{aligned}\left\langle (X_n/n\Delta t)^k \right\rangle &\sim \int_{-\pi}^{\pi} \left\{ |g_+(\phi)|^2 \left(-i\lambda'_+(\phi)/\lambda_+(\phi) \right)^k \right. \\ &\left. + |g_-(\phi)|^2 \left(-i\lambda'_-(\phi)/\lambda_-(\phi) \right)^k \right\} \frac{d\phi}{2\pi} \Delta t. \end{aligned} \tag{43}$$

This admits the following interpretation [25]: we can take as a random variable, a function Y from $\Omega = S^1 \times \{+,-\}$ to the reals, with $Y = -i\lambda'_+(\Phi)/\lambda_+(\Phi)$ on $S^1 \times \{+\}$, and $Y = -i\lambda'_-(\Phi)/\lambda_-(\Phi)$ on $S^1 \times \{-\}$. Here $\Phi : \Omega \to R$ is a random variable which projects on the circle S^1 with measure $|g_+(\phi)|^2 \Delta t \, (d\phi/2\pi)$ on $S^1 \times \{+\}$, and measure $|g_-(\phi)|^2 \Delta t (d\phi/2\pi)$ on

$S^1 \times \{-\}$. Since in the above limit, all the moments of $X_n/n\Delta t$ agree with all the moments of Y, and the support of $X_n/n\Delta t$ is compact, it follows that $X_n/n\Delta t$ converges weakly to Y. Hence we have that

$$\lim_{n\Delta t = t\to\infty} P(y_1 \le X_n/n\Delta t \le y_2) = P(y_1 \le Y \le y_2)$$
$$= \int_{T_+} |g_+(\phi)|^2 \,\Delta t\, \frac{d\phi}{2\pi} \qquad + \int_{T_-} |g_-(\phi)|^2 \,\Delta t\, \frac{d\phi}{2\pi}, \tag{44}$$

where the intervals of integration are $T_\pm = y_1 \le (-i\lambda'_\pm(\Phi)/\lambda_\pm(\Phi)) \le y_2$. Then it follows that in order to determine the long time position distribution we need only determine $g_\pm(\phi)$ and $\lambda_\pm(\phi)$.

Suppose now that we specialize to the case of a highly localized initial electron state [14] with $f(p) = f_\nu(p) = e^{-p^2/2\nu^2}/(\sqrt{\nu\sqrt{\pi}})$, where ν is large and positive and quantifies the extent of the localization of the Dirac particle's initial state — the larger is ν, the sharper is the initial localization. As ν approaches infinity one has that

$$c_+(x) \;\sim\; \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\nu}{\sqrt{\pi}}}\int_0^\infty e^{i\nu p x} e^{-p^2/2\nu^2}\, dp, \tag{45}$$
$$c_-(x) \;\sim\; \frac{i}{\sqrt{2\pi}}\sqrt{\frac{\nu}{\sqrt{\pi}}}\int_{-\infty}^0 e^{i\nu p x} e^{-p^2/2\nu^2}\, dp. \tag{46}$$

Note that in the limit $\nu \to \infty$, $\int_{-\infty}^{\infty} |c_+(x)|^2 dz = \int_{-\infty}^{\infty} |c_-(x)|^2 dz = 1/2$. If we now make $\nu\Delta t$ small by taking Δt small enough, then

$$\begin{aligned} g_\pm(\phi) \;&\sim\; i\sqrt{2\sqrt{\pi}/(\nu\Delta t^2)}\, e^{-\phi^2/(2\nu^2\Delta t^2)} f^*_{\pm+}(\phi) \qquad if \quad \phi > 0 \\ &\sim\; \sqrt{2\sqrt{\pi}/(\nu\Delta t^2)}\, e^{-\phi^2/(2\nu^2\Delta t^2)} f^*_{\pm-}(\phi) \qquad if \quad \phi < 0. \end{aligned} \tag{47}$$

Also

$$-\, i\lambda'_\pm(\phi)/\lambda_\pm(\phi) = \pm\frac{\sin\phi\cos\Delta t}{\sqrt{1-\cos^2\Delta t\cos^2\phi}} \equiv \pm h(\phi). \tag{48}$$

Note also that $|g_+(\phi)|^2 + |g_-(\phi)|^2 = 2\sqrt{\pi} e^{-\phi^2/\nu^2\Delta t^2}/(\nu\Delta t^2)$. To compute the asymptotic distribution we need to compute the integrals

$$\begin{aligned} I_1 \;&=\; \sum_i \int_{h_i^{-1}[y_1,y_2]} |g_+(\phi)|^2\,\Delta t \frac{d\phi}{2\pi} \\ &=\; \frac{1}{2\pi}\sum_i \int_{y_1}^{y_2} \frac{1}{\left|h'_+(h_i^{-1}(y))\right|} \left|g_+(h_i^{-1}(y))\right|^2 \Delta t\, dy \end{aligned} \tag{49}$$

and

$$
\begin{aligned}
I_2 &= \sum_i \int_{h_i^{-1}[-y_2,-y_1]} \left|g_-(\phi)\right|^2 \Delta t \frac{d\phi}{2\pi} \\
&= \frac{1}{2\pi} \sum_i \int_{y_1}^{y_2} \frac{1}{\left|h'_-(h_i^{-1}(-y))\right|} \left|g_-(h_i^{-1}(-y))\right|^2 \Delta t \, dy, \qquad (50)
\end{aligned}
$$

where the index i labels the local inverses of the function h. Because of (47), the only inverse relevant to leading order is the one that keeps ϕ close to zero. For this inverse, $h_i^{-1}(-y) = -h_i^{-1}(y)$. Furthermore, and realizing that $|v_\pm(\phi)\rangle^* = |v_\mp(-\phi)\rangle$, we can show that $|g_\pm(-\phi)|^2 = |g_\pm(\phi)|^2$. A direct computation then gives

$$
I_1 + I_2 = \frac{\Delta t \sin \Delta t}{2\pi} \int_{y_1}^{y_2} \frac{1}{(1-y^2)\sqrt{\cos^2 \Delta t - y^2}} \frac{2\sqrt{\pi}}{\nu \Delta t^2} e^{-\left(h_i^{-1}(y)\right)^2/\nu^2 \Delta t^2} \, dy. \qquad (51)
$$

If we set $y = h(\phi)$, then for all inverses ϕ_i we have $\cos^2 \phi_i$ =$(\cos^2 \Delta t - y^2)/\cos^2 \Delta t (1-y^2)$. Since for our inverse the value of ϕ_i is small we have

$$
\begin{aligned}
\left(h_i^{-1}(y)\right)^2 = \phi_i^2 \simeq \sin^2 \phi_i &= 1 - (\cos^2 \Delta t - y^2)/(\cos^2 \Delta t)(1-y^2) \\
&= y^2 \sin^2 \Delta t / \cos^2 \Delta t (1-y^2) \, . \qquad (52)
\end{aligned}
$$

Taking the limit $\Delta t \to 0$ we arrive at the asymptotic distribution function associated with the random variable $X_n / n\Delta t \sim Y$,

$$
\begin{aligned}
P(y_1 \le Y \le y_2) = \lim_{\Delta t \to 0} (I_1 + I_2) &= \int_{y_1}^{y_2} F(y) \, d, , \\
F(y) &= \frac{1}{\nu \sqrt{\pi}} \frac{1}{(1-y^2)^{3/2}} e^{-y^2/\nu^2(1-y^2)} \, . \qquad (53)
\end{aligned}
$$

In Fig. 1 we plot this two-horned probability distribution for three values of the localization parameter ν, and recognize it as the 1-dimensional analogue of the result obtained for the Dirac particle in three dimensions in [14], Eqn (3.1).

4 Discussion

It has been shown that the 1-dimensional Dirac evolution of a state with positive energy and definite spin is equivalent to a QRW, in the limit of

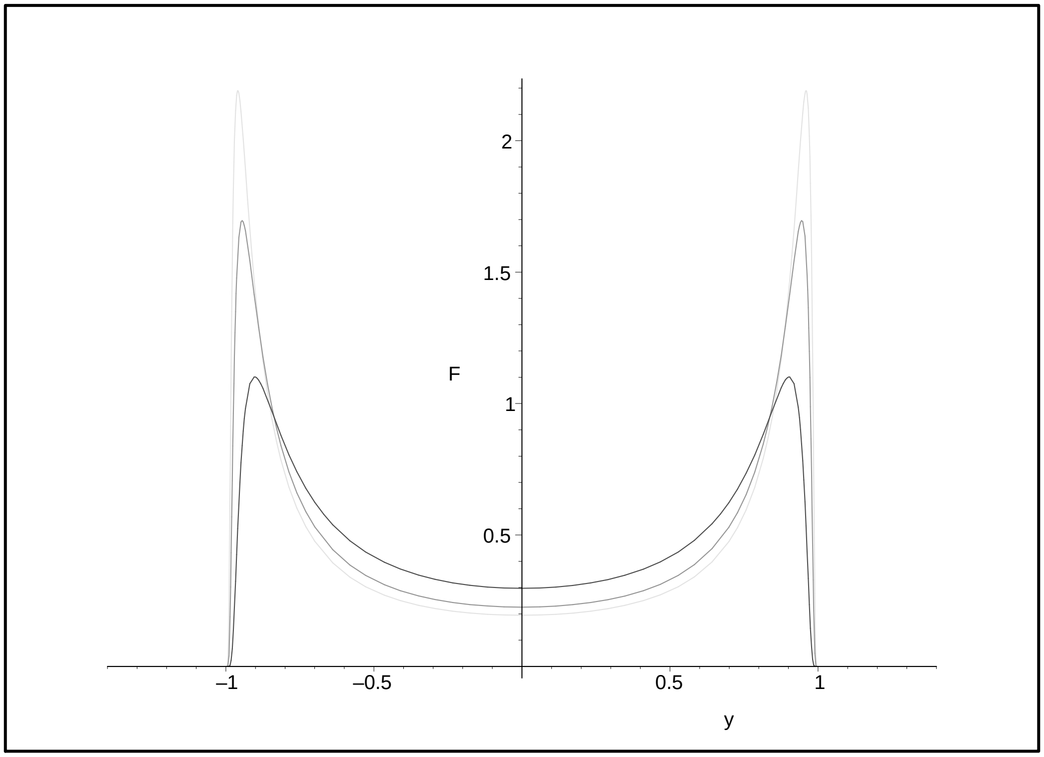

Figure 1: The asymptotic position probability density function, with localization parameter $\nu = 1.9$, 2.5 and 2.9. As ν increases, the plots become more sharply peaked near the ends of the interval.

small positional steps and a large number of iterations. An initial state that is highly localized, with all but one momentum component set to zero, spreads in the remaining direction at a speed that almost surely approaches the speed of light as the initial localization increases.

This relationship between the Dirac evolution and a QRW leads to the intriguing speculation that at some small space-time scale, there may really be a QRW defining the evolution of states of the relativistic electron, and that it is the Dirac evolution that is only a large scale approximation. One way to test this would be to make very precise measurements of the spreading characteristics of initially highly localized electron states over short distances. Comparison with the characteristics that are typical for a QRW, in particular the shape of the position probability distribution at early times, may reveal whether or not there is indeed a QRW underlying an approximate Dirac evolution.

It is tempting to speculate further that there may be some deep relationship between such an underlying QRW and the *Zitterbewegung* of the relativistic electron, as first discussed by Schrödinger [26]. This awaits further study.

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Figure caption:

Figure 1. The asymptotic position probability density function, with localization parameter $\nu = 1.9$, 2.5 and 2.9. As ν increases, the plots become more sharply peaked near the ends of the interval.